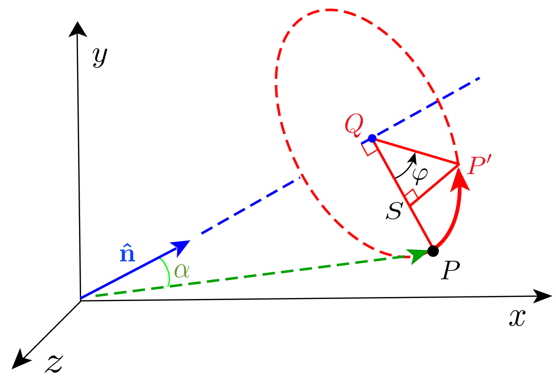

Figure 5.5: Rotation about an arbitrary axis given by the unit vector $\mathbf{\hat{n}}$.

5.3 Rotations about an arbitrary axis

Let us consider the rotation of a point P around an arbitrary axis of unit length $\mathbf{\hat{n}}$ to point P' (Figure 5.5). Point P' can be calculated as the result of translating the origin O of the coordinate system by vector $\overrightarrow{OP'}$, which in turn is the sum of three vectors:

$$\overrightarrow{OP'} = \overrightarrow{OQ} + \overrightarrow{QS} + \overrightarrow{SP'} \tag{5.2}$$

Here point Q is defined as the projection of point P on the axis with versor $\mathbf{\hat{n}}$, and point S as the projection of point P' on vector $\overrightarrow{QP}$. The three components of equation 5.2 can then be calculated as follows:

$$\overrightarrow{OQ} = (\overrightarrow{OP} \cdot \mathbf{\hat{n}})\mathbf{\hat{n}} \tag{5.3}$$

$$\overrightarrow{QS} = \overrightarrow{QP} \cos\varphi = (\overrightarrow{OP} - \overrightarrow{OQ}) \cos\varphi \tag{5.4}$$

$$\overrightarrow{SP'} = (\mathbf{\hat{n}} \times \overrightarrow{OP}) \sin\varphi \tag{5.5}$$

Equation 5.3 follows directly from the observation that vector $\overrightarrow{OQ}$ is the projection of vector $\overrightarrow{OP}$ on versor $\mathbf{\hat{n}}$.

To calculate $\overrightarrow{QS}$ we consider triangle $\triangle QSP'$. It is a right triangle, hence $|\overrightarrow{QS}| = |\overrightarrow{QP'}| \cos\varphi$. Rotation of a point around an axis does not change its distance from that axis, thus $|\overrightarrow{QP'}| = |\overrightarrow{QP}|$. In addition, Figure 5.5 shows that vector $\overrightarrow{QS}$ has the same direction as $\overrightarrow{QP}$. Putting these results together we obtain Equation 5.4.

To calculate $\overrightarrow{SP'}$ we consider triangle $\triangle QSP'$ again. Clearly, $|\overrightarrow{SP'}| = |\overrightarrow{QP'}| \sin\varphi = |\overrightarrow{QP}| \sin\varphi$. From the definition of cross product it follows that $|\overrightarrow{QP}| = |\mathbf{\hat{n}} \times \overrightarrow{OP}|$. Moreover, the product $\mathbf{\hat{n}} \times \overrightarrow{OP}$, being perpendicular to both of its arguments and obeying the right-hand rule, has the same direction as vector $\overrightarrow{SP'}$. By combining these results we obtain Equation 5.5.

Substituting Equations 5.3–5.3 into Equation 5.2, we get:

$$\begin{aligned}\overrightarrow{OP'} &= \overrightarrow{OQ} + (\overrightarrow{OP} - \overrightarrow{OQ})cos\varphi + (\hat{\mathbf{n}} \times \overrightarrow{OP})\sin\varphi \\ &= \overrightarrow{OQ}(1-\cos\varphi) + \overrightarrow{OP}cos\varphi + (\hat{\mathbf{n}} \times \overrightarrow{OP})\sin\varphi \\ &= (\overrightarrow{OP}\cdot\hat{\mathbf{n}})\cdot\hat{\mathbf{n}}(1-\cos\varphi) + \overrightarrow{OP}\cos\varphi + (\hat{\mathbf{n}} \times \overrightarrow{OP})\sin\varphi \qquad (5.6)\end{aligned}$$

This third equation is known as *Rodrigues' rotation formula*[1]. It can be conveniently used in its vector form, but matrix formulation is preferable when composing transformations. The matrix formulation is derived as follows:

$$\begin{aligned}(\overrightarrow{OP}\cdot\hat{\mathbf{n}})\cdot\hat{\mathbf{n}} &= (n_x x + n_y y + n_z z)\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} {n_x}^2 x + n_x n_y y + n_x n_z z \\ n_x n_y x + {n_y}^2 y + n_y n_z z \\ n_x n_z x + n_y n_z y + {n_z}^2 z \end{bmatrix} \\ &= \begin{bmatrix} {n_x}^2 & n_x n_y & n_x n_z \\ n_x n_y & {n_y}^2 & n_y n_z \\ n_x n_z & n_y n_z & {n_z}^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (5.7)\end{aligned}$$

$$\begin{aligned}(\hat{\mathbf{n}} \times \overrightarrow{OP}) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ n_x & n_y & n_z \\ x & y & z \end{vmatrix} = \begin{bmatrix} n_y z - n_z y \\ -n_x z + n_z x \\ n_x y - n_y x \end{bmatrix} \\ &= \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (5.8)\end{aligned}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit vectors on the X, Y and Z axes. By substituting Equations 5.7 and 5.8 into Equation 5.6, we obtain:

$$\begin{aligned}T &= \begin{bmatrix} {n_x}^2(1-\cos\varphi)+\cos\varphi & n_x n_y(1-\cos\varphi) - n_z\sin\varphi & n_x n_z(1-\cos\varphi) + n_y\sin\varphi \\ n_x n_y(1-\cos\varphi) + n_z\sin\varphi & {n_y}^2(1-\cos\varphi)+\cos\varphi & n_y n_z(1-\cos\varphi) - n_x\sin\varphi \\ n_x n_z(1-\cos\varphi) - n_y\sin\varphi & n_y n_z(1-\cos\varphi) + n_x\sin\varphi & {n_z}^2(1-\cos\varphi)+\cos\varphi \end{bmatrix} \\ &= \begin{bmatrix} {n_x}^2 + (1-{n_x}^2)\cos\varphi & n_x n_y(1-\cos\varphi) - n_z\sin\varphi & n_x n_z(1-\cos\varphi) + n_y\sin\varphi \\ n_x n_y(1-\cos\varphi) + n_z\sin\varphi & {n_y}^2 + (1-{n_y}^2)\cos\varphi & n_y n_z(1-\cos\varphi) - n_x\sin\varphi \\ n_x n_z(1-\cos\varphi) - n_y\sin\varphi & n_y n_z(1-\cos\varphi) + n_x\sin\varphi & {n_z}^2 + (1-{n_z}^2)\cos\varphi \end{bmatrix}\end{aligned}$$

[1]See https://en.wikipedia.org/wiki/Rodrigues%27_rotation_formula.