The Visual Computer

Hologram-like transmission of pictures

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A new digital representation of pictures is proposed. The main feature of this representation is: Given a string of data Z representing a picture with full resolution, various substrings of Z represent the same picture with appropriately lower resolution. This is analogous to a well-known property of holograms. The new representation is based on a particular picture traversal algorithm and uses overlapping sampling areas. The paper presents the principle of this representation, analyzes its overhead, and provides examples of picture reconstruction. An application of the hologram-like representation to the transmission of pictures with progressive resolution is indicated.

Key words: Progressive transmission – Sampling and reconstruction – Picture traversal

Numerous methods for the transmission of pictures have been studied in the past. In the simplest case, a picture is represented as an array of samples. Its transmission and display proceed along rows or columns, referred to as scan lines. If t_c is the time necessary to display the whole picture, in $1/k \cdot t_c$ time only one kth of the picture will be displayed. This part will be presented at full resolution. In some applications, for example slow transmission of visual information (videotex), or browsing through a database of stored images (Hill and Walker 1983), it may be desirable for the resolution of a picture, rather then its visible area, to increase while the transmission proceeds. To this end, methods of progressive transmission of pictures were developed (Pavlidis 1982). They use quad trees (Sloan and Tanimoto 1979) or binary trees (Knowlton 1980) as underlying data structures. Consequently, various initial substrings of the string of data representing the entire picture can be used for its reconstruction, with the resolution proportional to the length of the substring. However, non-initial substrings are meaningless.

This paper describes a new representation of pictures suitable for their transmission with progressive resolution. The main feature of this representation is: Given a string of data Z representing a picture with full resolution, various (not necessarily initial) substrings of Z represent the same picture with an appropriately lower resolution. This is analogous to the well-known feature of holograms: Any portion of a hologram represents the same picture as the whole hologram, but with a lower resolution. The new representation is based on a traversal algorithm (i.e. the selection and ordering of sampling points) with the following key property: Given a string of sampling points, $P = \langle p_0, p_1, p_2, \ldots \rangle$, various substrings of P consist of points uniformly distributed in the sampling region Q. Hence the longer a substring of sampled values is, the better the resolution of the reconstruction of the original picture can be achieved. An earlier version of this traversal algorithm was described in (Prusinkiewicz 1984).

This paper is organized as follows. In Sect. 2 the traversal algorithm is formally defined, and its essential properties are stated. A suitable sampling technique and the corresponding reconstruction method are described in Sect. 3. Section 4 provides an analysis of the overhead of the hologram-like representation. Section 5

The Visual Computer (1985) 1:185–193

generalizes the method to color pictures. A possible application is indicated in Sect. 6. The appendix contains proofs of the theorems.

1. Picture traversal algorithm

Intuitively, the traversal algorithm is based on two observations (Fig. 1):

- A translation of a set of sampling points uniformly distributed in a region of a plane is a set of uniformly distributed points;
- The union of appropriately translated sets of uniformly distributed points is also a set of uniformly distributed points, with a reduced distance between the adjacent points.

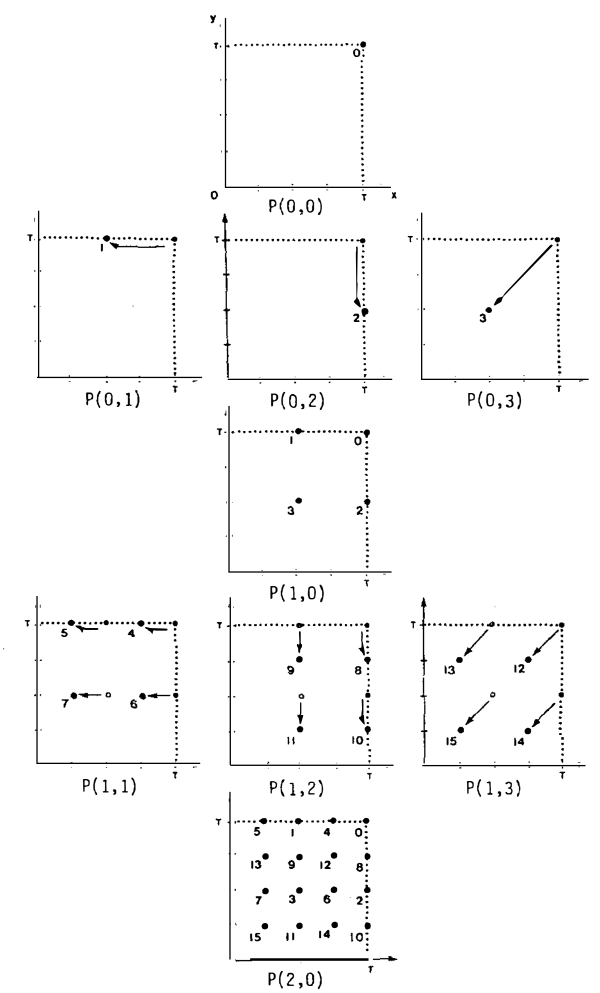


Fig. 1. Definition of the sequence of sampling points P

The string (sequence) of sampling points is defined recursively, by translating and concatenating previous strings. Consequently, various substrings consist of points uniformly distributed in the plane.

A formalization follows.

Let $P = \langle p_0, p_1, p_2, \ldots \rangle$ be a string of points in a plane. By $P(n,h)$ we denote the following substring of P:

$$P(n,h) = \langle p_{h \cdot 4^n}, p_{h \cdot 4^n + 1}, \ldots, p_{(h+1) \cdot 4^n - 1} \rangle$$

Furthermore, by $S(\langle p_0, p_1, \ldots, p_m \rangle, \vec{c})$ we denote translation of the substring $\langle p_0, p_1, \ldots, p_m \rangle$ by the vector $\vec{c}$:

$$S(\langle p_0, \ldots, p_m \rangle, \vec{c}) = \langle p_0 + \vec{c}, \ldots, p_m + \vec{c} \rangle$$

We will represent the translation vector $\vec{c}$ as $c_x \vec{1}_x + c_y \vec{1}_y$, where $\vec{1}_x$ and $\vec{1}_y$ are the unit vectors in the directions of axes x and y, respectively.

Definition 1. Let $T > 0$ denote the edge size of the square sampling region $Q(T)$ (its vertices are: $(0,0)$, $(0,T)$, (T,T), and $(T,0)$). The string of sampling points is then defined as follows:

$$P(0,0) = \langle p_0 \rangle = \langle T, T \rangle$$
$$P(n,1) = S(P(n,0), -T \cdot 2^{-n-1}\, \vec{1}_x)$$
$$P(n,2) = S(P(n,0), -T \cdot 2^{-n-1}\, \vec{1}_y)$$
$$P(n,3) = S(P(n,0), -T \cdot 2^{-n-1} (\vec{1}_x + \vec{1}_y))$$
$$P(n+1,0) = P(n,0) \circ P(n,1) \circ P(n,2) \circ P(n,3)$$

where $\circ$ denotes concatenation of strings, and $n = 0, 1, 2, \ldots$.

Theorem 1. *Let the binary word*

$$r_{2n-1} r_{2n-2} \ldots r_1 r_0$$

represent index k of a sample point $p_k = (x_k, y_k)$ in the pure binary number system:

$$k = \sum_{i=0}^{2n-1} r_i 2^i$$

The coordinates of point p_k are then equal to:

$$x_k = T\left(1 - \sum_{i=0}^{n-1} r_{2i} 2^{-(i+1)}\right)$$

$$y_k = T\left(1 - \sum_{i=0}^{n-1} r_{2i+1} 2^{-(i+1)}\right)$$

Proof in the Appendix.

Theorem 1 provides an explicit (non-recurrent) relationship between the index of a sampling point and its coordinates. This relation can also be used as a definition of sequence P (Prusinkiewicz 1984). It lacks, however, the intuitive flavor of Definition 1.
The central property of the string of sampling points P is given by Theorem 2. It refers to Definition 2 (Rosenfeld and Kak 1982), formalizing the notion of $m \times m$ points uniformly distributed in the square $Q(T)$.

Definition 2. Given a sampling region $Q(T)$, the sampling lattice of $m \times m$ elements, with the origin in point (c_x, c_y), is the set:

$$M_{m \times m}(c_x, c_y) = \left\{ \left(c_x + (i-1)\frac{T}{m}, c_y + (j-1)\frac{T}{m}\right) : i,j = 1,2,\dots,m \right\}$$

Theorem 2. *Let $\hat{P}(n,h)$ denote the (unordered) set of elements of the substring $P(n,h)$:*

$$\hat{P}(n,h) = \{p_{h \cdot 4^n}, p_{h \cdot 4^n + 1}, \dots, p_{(h+1)\cdot 4^n - 1}\}$$

For any n, $h = 0,1,2,\dots$, the set $\hat{P}(n,h)$ is a sampling lattice in the region $Q(T)$:

$$(\forall n, h = 0,1,2,\dots) \left(\exists c_x, c_y \in \left[0, \frac{T}{2^n}\right)\right)$$

$$\hat{P}(n,h) = M_{2^n \times 2^n}(c_x, c_y)$$

Proof in the Appendix.

2. Sampling and reconstruction

The sampling method used for the hologram-like transmission of pictures should make it possible to reconstruct a picture f from any number of uniformly distributed samples, with the resolution proportional to the number of samples considered.
The simplest sampling method – Shannon's sampling – does not meet this requirement. Since the number of samples which will actually be used to reconstruct picture f is not known when sampling, it is not possible to adequately filter out high frequency components of f. Consequently, a reconstruction of f from a small number of samples may be totally misleading, due to aliasing (Rosenfeld and Kak 1982).
Let us define the area sample z_k in point $p_k = (x_k, y_k)$ as the total amount of light which falls in the rectangle $(0,0)$, $(x_k, 0)$, (x_k, y_k), $(0, y_k)$:

$$z_k = \int_0^{x_k} \int_0^{y_k} f(x,y)\,dx\,dy \qquad (*)$$

A reconstruction of f can be performed, given area samples corresponding to any set of sampling points $\hat{P}(n,h)$. Following Theorem 1, the set $\hat{P}(n,h)$ forms the sampling matrix $M_{m \times m}(c_x, c_y)$ for some m, c_x, c_y. Consequently, each sampling point p_k can be expressed as:

$$\left(c_x + (i-1)\frac{T}{m}, c_y + (j-1)\frac{T}{m}\right)$$

where $i,j \in \{1,2,\dots,m\}$. We will use i and j as indices, and write $p_{i,j}$ instead of p_k. Similarly, if $p_{i,j} = p_k$, we will write $z_{i,j}$ instead of z_k. By referring to the definition of the area sample, we then obtain:

$$\frac{m^2}{T^2} \int_{x_i}^{x_i + \frac{T}{m}} \int_{y_i}^{y_i + \frac{T}{m}} f(x,y)\,dx\,dy$$

$$= \frac{m^2}{T^2}(z_{i+1,j+1} - z_{i+1,j} - z_{i,j+1} + z_{i,j})$$

where $i,j = 1,2,\dots,m-1$ (an extension to $i=0$ and $j=0$ is straightforward). The above equation is the basis of picture reconstruction. The left side represents the average value of function f (average gray level) in the square $Q_{i,j}$ with vertices:

$$(x_i, y_j),\ (x_{i+\frac{T}{m}}, y_j),\ (x_{i+\frac{T}{m}}, y_{j+\frac{T}{m}}),\ (x_i, y_{j+\frac{T}{m}})$$

This value (known as standard sample) can be directly used as an approximation of f in $Q_{i,j}$. Due to averaging, the reconstruction of f based on standard samples will be automatically antialiased.
Examples of the reconstruction of pictures from their hologram-like representations are shown in Figs. 2 and 3. These figures were obtained on a laser printer, with an 8×8 dither matrix (Foley and van Dam 1982) used to simulate 64 gray levels.

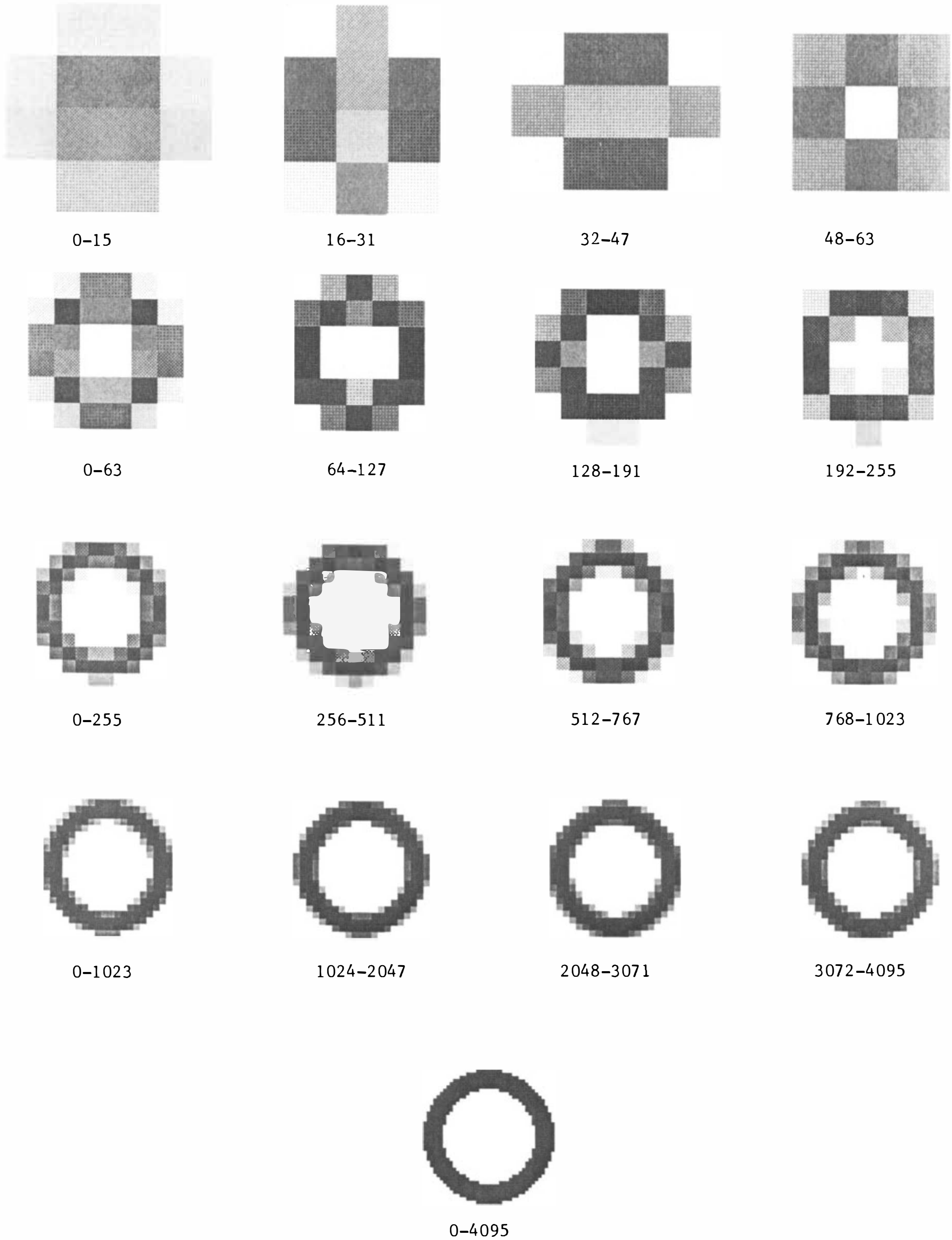
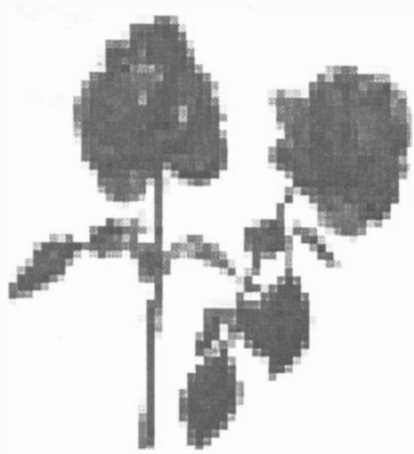


Fig. 2. Example of the hologram-like transmission of a picture. Numbers indicate indices of area samples z_k used for reconstruction

0–4095

4096–8191

8192–12287



12288–16383

0–16383

Fig. 3. Example of the hologram-like transmission of a picture. Numbers indicate indices of area samples z_k used for reconstruction

3. Analysis of overhead

Suppose that the original picture is sampled using a lattice of $2^n \times 2^n$ points. Furthermore, suppose that the gray level function f takes values from the set $\{0, 1, \dots, 2^d - 1\}$ (after quantization). From the formula (*) it follows that the area sample $z_{i,j}$ in point $p_{i,j}$ can take any value from 0 to $i \cdot j \cdot (2^d - 1)$. The number of bits necessary to represent this sample is equal to:

$$\lceil \log_2(i \cdot j \cdot (2^d - 1)) \rceil \approx d + \lceil \log_2(i) + \log_2(j) \rceil$$

where $\lceil x \rceil$ denotes the ceiling function. Consequently, the total length of the hologram-like representation of f (in bits) is equal to:

$$K_1 = 4^n d + \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} \lceil \log_2(i) + \log_2(j) \rceil$$

Since the number of bits required to send 4^n d-bit samples is equal to $4^n \cdot d$, the overhead related to the hologram-like representation can be expressed as:

$$O_1 = \frac{K_1 - 4^n \cdot d}{4^n \cdot d}$$

Value K_1 is calculated under the assumption that representations of area samples $z_{i,j}$ may have variable lengths. Decoding of the picture can actually be simplified if all samples are represented by words of equal length: $d + 2n$. The length of the hologram-like representation is then equal to $K_2 = 4^n \cdot (d + 2n)$. The corresponding overhead is equal to:

$$O_2 = \frac{K_2 - 4^n \cdot d}{4^n \cdot d} = 2\frac{n}{d}$$

The overheads O_1 and O_2 calculated for various values of $N = 2^n$ and d are shown in Fig. 4. The overheads are approximately proportional to the logarithm of the number of samples n, and inversely proportional to the number of bits per standard sample d. The variable-length representation of samples does not significantly reduce the overhead.

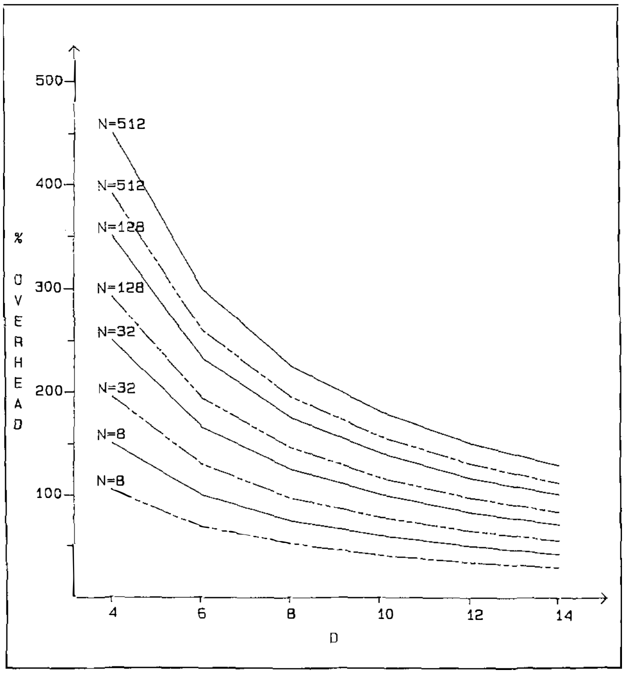


Fig. 4. Overhead of the hologram-like representation of pictures. N indicates the total number of area samples taken (N^2). D is the number of bits per area sample (pixel). Dashed lines correspond to the variable-length representation of samples (overhead O_1). Continuous lines correspond to the fixed-length representation (overhead O_2)

4. Transmission of color pictures

While the hologram-like representation deals essentially with one-dimensional data, its generalizations to color pictures can be sought. Various color models and methods for mapping multidimensional color data to one-dimensional data stream can be assumed. Figure 5 shows an example of the hologram-like transmission based on the RGB color space. Each sample is a triplet of the corresponding RGB values. During reconstruction, the color planes are processed separately and the results are blended together. Consequently, color interpolation in the RGB space occurs.

5. Concluding remarks

A new method for the transmission of pictures has been proposed. The main property of this method is: given a data string Z representing a picture f with full resolution, various substrings of Z represent f with a resolution proportional to the length of the substring. Analysis of the overhead of the method is given. Examples of

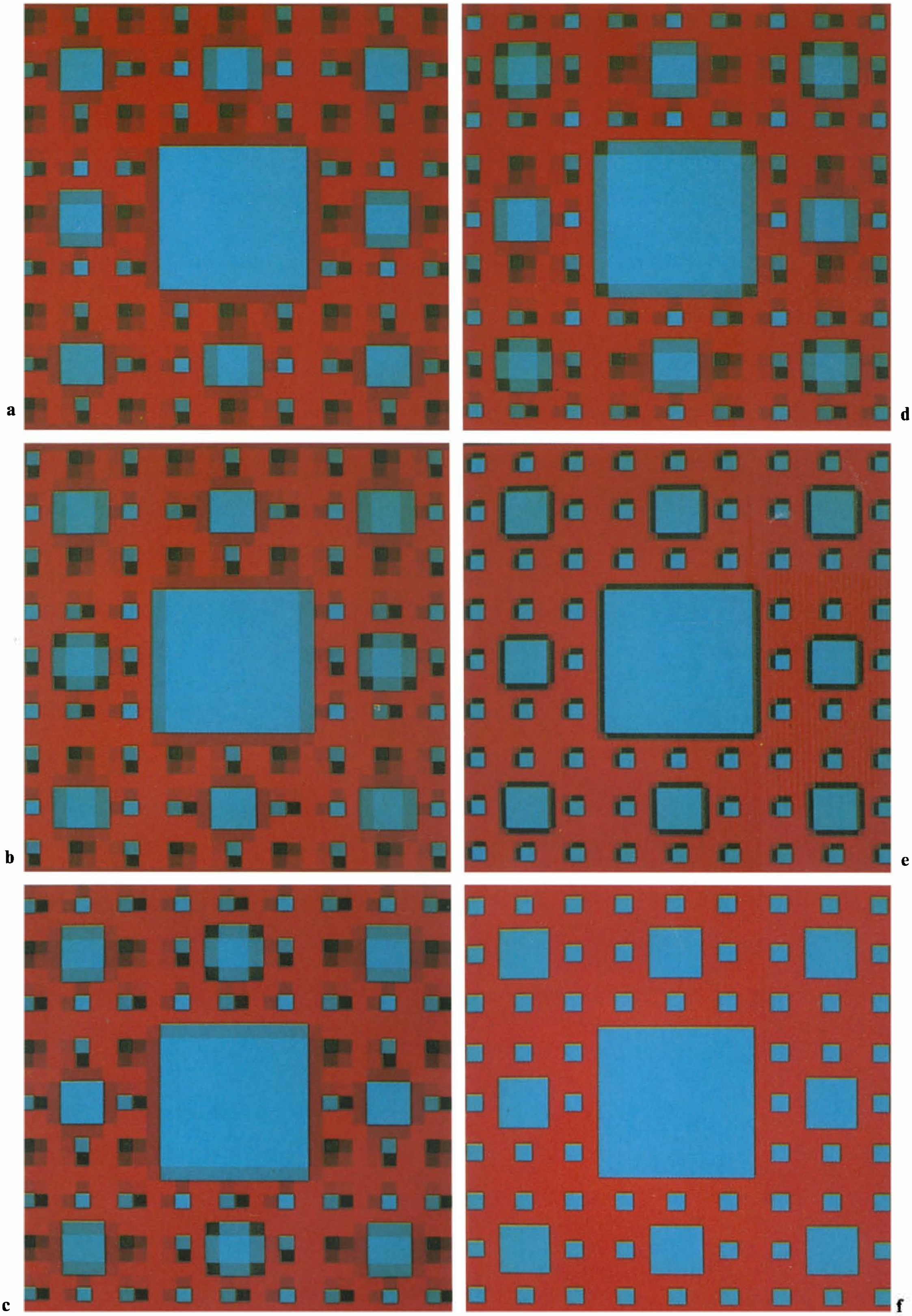


Fig. 5a–f. Example of the hologram-like transmission of a picture. Reconstruction from samples: **a** 0–1023, **b** 1024–2047, **c** 2048–3071, **d** 3072–4095, **e** 0–4095, **f** 0–16383 (original picture)

the reconstruction of pictures from the hologram-like representation are presented.
The method is applicable, for example, to browsing through a set of pictures sent round robin over a communications channel (in a videotex system). Pictures can be quickly reconstructed from a small number of samples, allowing for previewing before the full resolution reconstruction of the selected picture proceeds.

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Appendix: proofs of the theorems

Proof of Theorem 1.

Let $L(R)$ denote the number represented by the binary word R in the pure binary number system. Specifically, the numerical value of the empty word λ is zero: $L(\lambda)=0$. As usual, point (.) can be used to separate the fractional part of R from its integer part. Using this notation, Theorem 1 can be restated as follows:

Let the word $b_{n-1}a_{n-1}\dots b_0 a_0$ represent index k in the pure binary number system: $k=L(b_{n-1}a_{n-1}\dots b_0 a_0)$. The point q_k with coordinates:

$$x_k=T(1-L(.a_0\dots a_{n-1}))$$
$$y_k=T(1-L(.b_0\dots b_{n-1}))$$

coincides with point p_k specified by Definition 1.
Proof by induction on n.
– For $n=0$ the word $b_{n-1}a_{n-1}\dots b_0 a_0$ is empty. Consequently, $k=L(\lambda)=0$, $x_0=T(1-L(\lambda))=T$, and $y_0=T(1-L(\lambda))=T$. Thus, $q_0=(T,T)$ is equal to p_0.
– Suppose the theorem true for $n\geqq 0$. Thus, $q_k=p_k$ for any k represented as $b_{n-1}a_{n-1}\dots b_0 a_0$. For $n+1$ four cases can be distinguished: $b_n a_n=00, 01, 10$ and 11. The case $b_n a_n=00$ is trivial because it leads to a previously considered value of k. (The coordinates x_k and y_k are not affected by trailing zeroes in their binary representations.) In the next case ($b_n a_n=01$) index $k'=L(01\,b_{n-1}a_{n-1}\dots b_0 a_0)$ can be represented as $k'=4^n+k$, where $k=L(b_{n-1}a_{n-1}\dots b_0 a_0)$. Point $q_{k'}$ has coordinates:

$$x_{k'}=T(1-L(.a_0\dots a_{n-1}1))=x_k-T\cdot 2^{-n-1}$$
$$y_{k'}=T(1-L(.b_0\dots b_{n-1}0))=y_k$$

Or, $q_{k'}=q_k-T\cdot 2^{-n-1}\vec{1}_x$. On the other hand, from the Definition 1 it follows that $p_{k'}=p_k-T\cdot 2^{-n-1}\vec{1}_x$. Since $q_k=p_k$ (by the inductive hypothesis), $q_{k'}=p_{k'}$ as well. In the remaining cases ($b_{n+1}a_{n+1}=10$ and 11) the equality $q_{k'}=p_{k'}$ can be proved the analogous way. Thus, the theorem is true for all $n=0,1,2,\dots$. □

Proof of Theorem 2.
– Let us first consider the case $h=0$. Following Theorem 1, the set $\hat{P}(n,0)$ consists of all points p_k with coordinates:

$$x_k=T(1-L(.a_0\dots a_{n-1}))$$
$$y_k=T(1-L(.b_0\dots b_{n-1}))$$

It is known that the 2^n numbers represented by the words $.a_0\dots a_{n-1}$ form the arithmetic sequence with the first element 0 and the difference 2^{-n}. Naturally, the 2^n numbers represented by the words $.b_0\dots b_{n-1}$ form the same sequence. Hence,

$$\hat{P}(n,0)=\{(T(1-i\cdot 2^{-n}),\,T(1-j\cdot 2^{-n})):$$
$$i,j=0,1,\dots,2^n-1\}=M_{2^n\times 2^n}(2^{-n},2^{-n})$$

– For $h>0$ index k' of any element of the set $\hat{P}(n,h)$ can be expressed as $k'=h\cdot 4^n+k$, where $k\in\{0,1,\dots,4^n-1\}$. Consequently, the binary representation of k' can be written as:

$$b_{s-1}a_{s-1}\dots b_n a_n b_{n-1}a_{n-1}\dots b_0 a_0$$

where $h=L(b_{s-1}a_{s-1}\dots b_n a_n)$ and

$k=L(b_{n-1}a_{n-1}\dots b_0 a_0)$.

As a result, the coordinates of point $p_{k'}$ can be expressed as:

$$x_{k'}=T(1-L(.a_0\dots a_{n-1}))-T\cdot 2^{-n}L(.a_n\dots a_{s-1})$$
$$y_{k'}=T(1-L(.b_0\dots b_{n-1}))-T\cdot 2^{-n}L(.b_n\dots b_{s-1})$$

Thus, $p_{k'}=p_k-c_x\vec{1}_x-c_y\vec{1}_y$, where c_x and c_y are constants dependent only of h. Consequently, $\hat{P}(n,h)$ is a translation of the sampling lattice $\hat{P}(n,0)$, or $\hat{P}(n,h)$ is also a simpling lattice. □

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