Lane-Riesenfeld algorithm ("The chasing game")

Note: For simplicity, the rules below deal with vertices (points) only.
Rules dealing also with edges are in the file
... Lecture 20/CPSC 453 - Subdivision - 2019.pdf

Rule $p: \; P(v) \to P(v)\,P(v)$ // duplicate point $P(v)$

$q: \; P(v) > P(v_R) \to P\left(\tfrac{1}{2}v + \tfrac{1}{2}v_R\right)$ // "chase right neighbor."

Example: We begin with the initial polygon:

$$P(v_1) \quad P(v_2) \quad P(v_3) \quad P(v_4)$$

circular word

p: $P(v_1)\,P(v_1)\;\; P(v_2)\,P(v_2)\;\; P(v_3)\,P(v_3)\;\; P(v_4)\,P(v_4)$

q: $P(v_1)\,P\left(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2\right)P(v_2)\,P\left(\tfrac{1}{2}v_2+\tfrac{1}{2}v_3\right)P(v_3)\,P\left(\tfrac{1}{2}v_3+\tfrac{1}{2}v_4\right)P(v_4)\,P\left(\tfrac{1}{2}v_4+\tfrac{1}{2}v_1\right)$

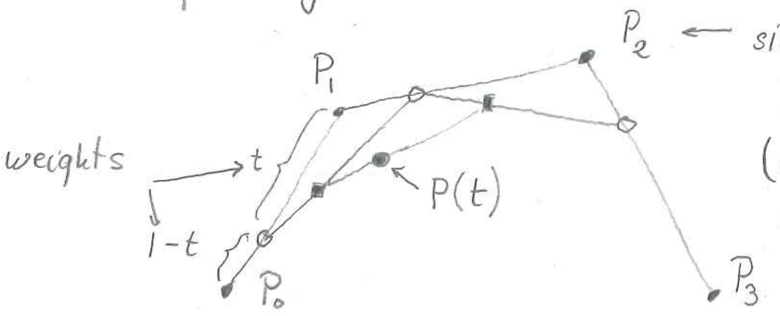
q: $P\left(\tfrac{3}{4}v_1+\tfrac{1}{4}v_2\right)P\left(\tfrac{1}{4}v_1+\tfrac{3}{4}v_2\right)P\left(\tfrac{3}{4}v_2+\tfrac{1}{4}v_3\right)P\left(\tfrac{1}{4}v_2+\tfrac{3}{4}v_3\right)P\left(\tfrac{3}{4}v_3+\tfrac{1}{4}v_4\right)P\left(\tfrac{1}{4}v_3+\tfrac{3}{4}v_4\right)P\left(\tfrac{3}{4}v_4+\tfrac{1}{4}v_1\right)P\left(\tfrac{1}{4}v_4+\tfrac{3}{4}v_1\right)$

q: $P\left(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2\right)P\left(\tfrac{1}{8}v_1+\tfrac{3}{4}v_2+\tfrac{1}{8}v_3\right)P\left(\tfrac{1}{2}v_2+\tfrac{1}{2}v_3\right)P\left(\tfrac{1}{8}v_2+\tfrac{3}{4}v_3+\tfrac{1}{8}v_4\right)\ldots$

Cycle (pq^2) repeated: Chaikin's corner-cutting algorithm
(= quadratic B-spline)

Cycle (pq^3) repeated: cubic B spline

De Casteljau algorithm



← simplified notation:
$P_2 = P(v_2)$
(it is possible as long as we are not talking about context)

Like subdivision algorithms (including the "chasing game"), the de Casteljau algorithm is a geometric construction. Unlike the subdivision algorithms, however, it returns only a single point for a given value of parameter t (the entire curve is obtained by sweeping t from 0 to 1).

We can also find point P(t) algebraically. This provides a conceptual bridge to the algebraic description of parametric curves

Let $a=1-t$, $b=t$. Then point P(t) can be calculated as follows

$$P_0 \qquad P_1 \qquad P_2 \qquad P_3$$

$$aP_0+bP_1 \qquad aP_1+bP_2 \qquad aP_2+bP_3$$

$$a(aP_0+bP_1)+b(aP_1+bP_2) \qquad a(aP_1+bP_2)+b(aP_2+bP_3)$$

$$\| \qquad\qquad \|$$

$$a^2P_0+2abP_1+b^2P_2 \qquad a^2P_1+2abP_2+b^2P_3$$

$$a(a^2P_0+2abP_1+b^2P_2)+b(a^2P_1+2abP_2+b^2P_3)$$

$$\|$$

$$a^3P_0+3a^2bP_1+3ab^2P_2+b^3P_3$$

$a=1-t$
$b=t$
$n=3$

$$P(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i P_i$$

From Bernstein polynomials to the (cubic) Bezier matrix

Recall: $P(t) = \sum_{i=0}^{n} \underbrace{\binom{n}{i}(1-t)^{n-i} t^i}\, P_i$

these are:
- Bernstein polynomials
- blending functions

They add up to 1 ← why is this important?

proof

$$\sum_{i=0}^{n} \binom{n}{i}(1-t)^{n-i} t^i = ((1-t)+t)^n = 1^n = 1$$

Let's evaluate $P(t)$ for $n=3$ (practically important special case)

$$P(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3 =$$

$$= (1-3t+3t^2-t^3)P_0 + (3t-6t^2+3t^3)P_1 + (3t^2-3t^3)P_2 + t^3 P_3 =$$

$$= \begin{bmatrix} -t^3+3t^2-3t+1 & 3t^3-6t^2+3t & -3t^3+3t^2 & t^3 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} =$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\text{Bezier matrix}} \underbrace{\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}}_{\text{Bezier geometry vector}}$$

Connecting Bézier curves

Consider two Bézier curves, specified by control polygons $P_0, \ldots, P_3$ and $Q_0, \ldots, Q_3$, respectively. From the de Casteljau algorithm we know that Bézier curves go through the initial point and the endpoint of the control polygons. Thus, the condition for C^0 continuity between the two curves is $P_3 = Q_0$.

What condition must the control polygons satisfy to provide C^1 continuity at the junction? To answer this question, let's calculate the derivative of the first curve at P_3 (i.e, for $t=1$), and for the second curve at Q_0 (i.e, for $t=0$

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}, \quad \text{thus}$$

$$P'(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 3t^2+6t-3 & 9t^2-12t+3 & -9t^2+6t & 3t^2 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

thus $P'(0) = -3P_0 + 3P_1 = 3(P_1 - P_0)$

$P'(1) = -3P_2 + 3P_3 = 3(P_3 - P_2)$

Likewise, $Q'(0) = 3(Q_1 - Q_0)$

$Q'(1) = 3(Q_3 - Q_2)$

Condition for C^1 continuity:

$$\boxed{P_3 - P_2 = Q_1 - Q_0}$$

Exercises

1) What is the difference between parametric continuity (C^n) and geometric continuity (G^n)?

2) What is the condition for the two Bézier curves specified by control polygons $P_0, \ldots, P_3$ and $Q_0, \ldots, Q_3$ to be G^1 continuous at their junction? What is the condition for them to be C^2 continuous?

Cubic B-splines viewed as parametric curves

In a manner similar to the Bézier curves, a (segment of) a B-spline curve is defined as

$$P(t) = T M_S G_S$$

However, the B-spline geometry matrix is now different:

$$M_S = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

A segment of a B-spline is defined as

$$P(t) = [t^3 \; t^2 \; t \; 1] \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix}$$

Likewise, next segment is defined as

Notice that consecutive control polygons partially overlap.

$$P'(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_i \\ P_{i+1} \\ P_{i+2} \\ P_{i+3} \end{bmatrix}$$

Let's verify C^0 continuity at the junction

$$P(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix} = \frac{P_i + 4P_{i+1} + P_{i+2}}{6}$$

$$P'(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_i \\ P_{i+1} \\ P_{i+2} \\ P_{i+3} \end{bmatrix} = \frac{P_i + 4P_{i+1} + P_{i+2}}{6}$$

same thing $\Downarrow$ C^0 continuity

Exercise

Prove that B-spline curves are also C^1 and C^2-continuous at the junction

From Bézier curves to Bézier surfaces

Recall that a Bézier curve can be described as

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} M_B \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Bezier matrix

Bézier geometry vector.

We can extend this formula to describe 4 Bézier curves at once:

$$\begin{bmatrix} P_0(t) & P_1(t) & P_2(t) & P_3(t) \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} M_B \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix}$$

We can also transpose this equation:

$$\begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} = \begin{bmatrix} P_{00} & P_{10} & P_{20} & P_{30} \\ P_{01} & P_{11} & P_{21} & P_{31} \\ P_{02} & P_{12} & P_{22} & P_{32} \\ P_{03} & P_{13} & P_{23} & P_{33} \end{bmatrix} M_B^T \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

we later refer to this vector as T^T

We now consider these points as the vertices of a control polygon that defines another Bézier curve (of parameter s), thus obtaining

$$P(s,t) = \underbrace{\begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix}}_{S} M_B \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} = S\, M_B \begin{bmatrix} P_{00} & P_{10} & P_{20} & P_{30} \\ P_{01} & P_{11} & P_{21} & P_{31} \\ P_{02} & P_{12} & P_{22} & P_{32} \\ P_{03} & P_{13} & P_{23} & P_{33} \end{bmatrix} M_B^T\, T^T$$

We can construct other bicubic surfaces (e.g. B-spline) in a similar way